Edge Preserving Image Denoising in Reproducing Kernel Hilbert Spaces

A novel approach for removing any type of additive noise from a grayscale image

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Outline

1. Image Denoising
   - The problem
   - Typical Solutions
2. Reproducing Kernel Hilbert Spaces
   - Definition and Main Properties of RKHS
   - Why RKHS?
   - Two Important Theorems
3. Kernelised Noise Removal
   - Basic Idea
   - Formulation
   - Parameter Selection
   - Experiments

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- The objective of the image denoising problem is to estimate the original image f from the noisy one $\hat{f}$.

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2. Impulse Noise: $p(z) = P_a$, if $z = a$, $p(z) = P_b$, if $z = b$, $p(z) = 0$ otherwise.

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- The idea is to express f as a span of some base functions f_i.
- We choose the base functions f_i to belong to a RKHS.

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Consider a linear class $\mathcal{H}$ of real valued functions f defined on a set $\mathcal{X}$ (in particular $\mathcal{H}$ is a Hilbert space) for which there exists a function $\kappa : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ with the following two properties:

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1. For every $x \in \mathcal{X}$, $\kappa(x, \cdot)$ belongs to $\mathcal{H}$.
2. κ has the so called reproducing property, i.e.,

$$f(x) = \langle f, \kappa(x, \cdot) \rangle_{\mathcal{H}}, \text{ for all } f \in \mathcal{H}, \; x \in \mathcal{X}, \tag{1}$$

in particular $\kappa(x, y) = \langle \kappa(x, \cdot), \kappa(y, \cdot) \rangle_{\mathcal{H}}$.

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  1. Map the finite dimensionality input data from the input space $\mathcal{X}$ into a higher dimensionality (possibly infinite) RKHS $\mathcal{H}$.
  2. Perform a linear processing on the mapped data in $\mathcal{H}$.

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- "Given an algorithm which is formulated in terms of an inner product, one can construct an alternative algorithm by replacing the inner product with a positive kernel κ".

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- **Inhomogeneous Polynomial Kernel** $\kappa(x, y) = (\langle x, y \rangle + c)^d$
- **B_n-Spline of odd order Kernel** $\kappa(x, y) = B_{2r+1}(\|x - y\|)$, with $B_n = \bigotimes_{i=1}^{n} I_{[-\frac{1}{2},\frac{1}{2}]}$

Outline

The Representer Theorem

Theorem

Denote by $\Omega : [0, \infty) \to \mathbb{R}$ *a strictly monotonic increasing function, by* $\mathcal{X}$ *a set and by* $c : (\mathcal{X} \times \mathbb{R}^2)^m \to \mathbb{R} \cup \{\infty\}$ *an arbitrary loss function. Then each minimizer* $f \in \mathcal{H}$ *of the regularized risk functional*

$$c\left((x_1, y_1, f(x_1)), \ldots, (x_N, y_N, f(x_N)\right) \quad + \quad \Omega\left(\|f\|_{\mathcal{H}}\right)$$

admits a representation of the form

$$f(x) = \sum_{n=1}^{N} \alpha_n \kappa(x_n, x).$$

Example of the Representer Theorem

Consider the problems

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In both cases the minimizer admits the form:

$$f(x) = \sum_{n=1}^{N} \alpha_n \kappa(x_n, x).$$

The semi-parametric Representer Theorem

Theorem

Suppose that in addition to the assumptions of the previous theorem we are given a set of M real-valued functions $\{\psi_p\}_{p=1}^M : \mathcal{X} \to \mathbb{R}$, with the property that the $N \times M$ matrix $(\psi_p(x_n))_{n,p}$ has rank M. Then any $f := \tilde{f} + h$, with $\tilde{f} \in \mathcal{H}$ and $h \in span\{\psi_p\}$, minimizing the regularized risk functional

$$c\left((x_1, y_1, f(x_1)), \ldots, (x_N, y_N, f(x_N)\right) \quad + \quad \Omega\left(\|\tilde{f}\|_{\mathcal{H}}\right)$$

admits a representation of the form

$$f(x) = \sum_{n=1}^{N} \alpha_n \kappa(x_n, x) + \sum_{p=1}^{M} \beta_p \psi_p(x).$$

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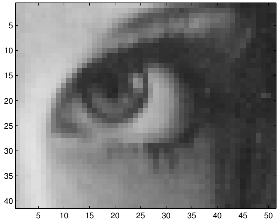
- Evidently, one cannot effectively approximate a non-smooth function f as a span of base functions of a specific RKHS.
- The semi-parametric Representer Theorem, may be used to impose non-smoothness through the functions ψ_p.

Outline

An image is actually a function



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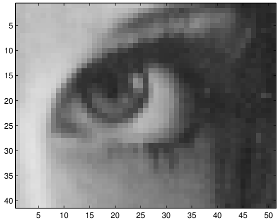

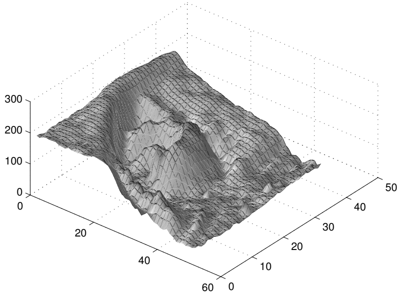

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Rectangular area neighborhood

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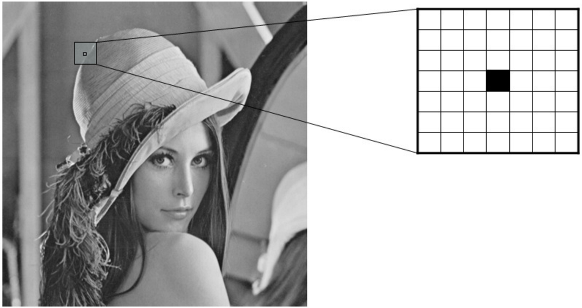

Choosing functions to represent edges

- Let $\hat{f}$ be the given "noisy" neighborhood of one pixel with dimensions $N \times M$, i.e. the $\hat{z}_{m,n} = \hat{f}(x_m, y_n)$ for $m = 1, \ldots, M$, $n = 1, \ldots, N$, are the given pixel values of the noisy neighborhood.

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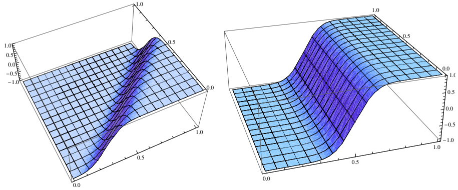

The expansion

- Next, we assume for the denoised image f that $f \in \mathcal{F} = \mathcal{H} \; + \; h_0 \boldsymbol{I} \; + \; \mathrm{span}\{\psi_1, \ldots, \psi_K\}$ (where $\boldsymbol{I} \in \mathbb{R}$ stands for the constant function i.e. $\boldsymbol{I}(x, y) = 1$).

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  (where $\boldsymbol{I} \in \mathbb{R}$ stands for the constant function i.e. $\boldsymbol{I}(x, y) = 1$).
- Hence f admits the form

$$f = \tilde{f} + h_0 \boldsymbol{I} + \sum_{k=1}^{K} \beta_k \psi_k.$$

Optimization

- We solve the following minimization problem for each pixel (using Polyak's Projected Subgradient Method):

$$\underset{f \in \mathcal{F}}{\text{minimize}} \quad \sum_{m=1}^{M} \sum_{n=1}^{N} |f(x_m, y_n) - \hat{z}_{m,n}| \quad + \quad \frac{\lambda}{2} \|\tilde{f}\|_{\mathcal{H}}^2 \quad + \quad \frac{\mu}{2} \sum_{k=1}^{K} |\beta_k|^2,$$

where $\tilde{f}$ is the part of the expansion of f that lives on $\mathcal{H}$.

Semiparametric Representer Theorem

Applying a version of the semiparametric Representer Theorem we take that f admits the form

$$f = \sum_{m=1}^{M} \sum_{n=1}^{N} \alpha_{m,n} \kappa\left((x_m, y_n), (\cdot, \cdot)\right) + h_0 \boldsymbol{I} + \sum_{k=1}^{K} \beta_k \psi_k.$$

Remarks

$$\operatorname*{minimize}_{f\in\mathcal{F}} \quad \sum_{m=1}^{M}\sum_{n=1}^{N}|f(x_m, y_n) - \hat{z}_{m,n}| \quad + \quad \frac{\lambda}{2}\|\tilde{f}\|_{\mathcal{H}}^2 \quad + \quad \frac{\mu}{2}\sum_{k=1}^{K}|\beta_k|^2,$$

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- Note that the use of l_2 instead of the l_1 norm in the cost function would make the method sensitive to outliers (e.g., impulses).

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- Note that the use of l_2 instead of the l_1 norm in the cost function would make the method sensitive to outliers (e.g., impulses).
- Furthermore, the l_1 norm adds some sort of sparsity to the expansion.
- For even more sparse solutions, one may also adopt the l_1 norm for the regularization terms.

Remarks

In the case of the Gaussian Kernel:

$$\|\tilde{f}\|_{\mathcal{H}} = \int_{\mathcal{X}} \sum_{n} \frac{\sigma^{2n}}{n!2^n} (O^n \tilde{f}(x))^2 dx,$$

with $O^{2n} = \Delta^n$ and $O^{2n+1} = \nabla \Delta^n$, Δ being the Laplacian and ∇ the gradient operator.
Thus, we see that the regularization term $\|\tilde{f}\|_{\mathcal{H}}^2$ "penalizes" the derivatives of the minimizer's part that lives on $\mathcal{H}$.

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  - For "steeper" edges, the value of μ is smaller.

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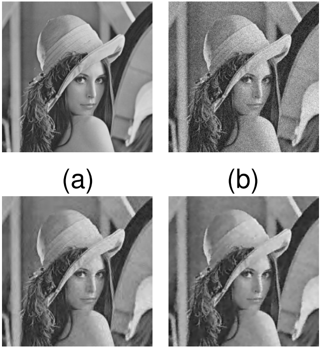
Gaussian Noise Removal



(a) (b)

(c) (d)

Figure: (a) Original Image, (b) Original with additive Gaussian Noise - PSNR=18,7146 dB, (c) Wavelet Denoising (BiShrink) - PSNR=29,3536 dB, (d) Kernelised Denoising - PSNR=29,4535 dB

Impulse Noise Removal

(a) (b)
(c) (d)

Figure: (a) Original Image, (b) Original with additive Impulse Noise - PSNR=12,7562 dB, (c) Wavelet Denoising - PSNR=25,2574 dB, (d) Kernelised Denoising - PSNR=30,1146 dB

Mixed Noise Removal

(a) (b)

Figure: (a) Image with additive mixed Noise (Gaussian + Impulse) - PSNR=21 dB, (b) Kernelised Denoising - PSNR=32,28 dB

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Advantages of the kernel based methodology:

- Independence of the noise statistics.
- Superior results in the presence of impulse or mixed noise.
- In the presence of gaussian noise, the kernel based method gives results similar to wavelet-based techniques that require no additional information for the noise statistics (such as BiShrink).

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- In the presence of gaussian noise, the cutting edge wavelet-based methods (such as BM3D, BLS-GSM), which require some sort of knowledge of the standard deviation σ, give superior results.

Future Research

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- Applying the kernel-based approach in the context of super-resolution.